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Expansive Maps in Multiplicative Generalised Metric Space

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Abstract—*In this paper, first we introduce the notion of multiplicative generalised metric spaces and expansive maps in this space. We prove some fixed point theorems for expansive maps in setting of newly defined spaces. We also provide some examples in support of our results.*

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1. INTRODUCTION

In 2007, Bashirov[1] defined multiplicative calculus and gave the notion of multiplicative metric spaces follows.

Let X be a nonempty set. Multiplicative metric[1] is a mapping d: $X\times X\to R$ satisfying the following conditions:
(m1) $d(x, y) > 1$ for all $x, y \in X$ and $d(x, y) =1$ if and only if $x=y$,
(m2) $d(x, y)=d(y, x)$ for all $x, y \in X$,
(m3) $d(x, z) \leq d(x, y) .d(y, z)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

The pair (X, d) is a multiplicative metric space.
For more detail on multiplicative metric spaces one can refer to [7].
In 2006, Mustafa and Sims[6] introduce the notion of G-metric space as follows.
Let X be a non empty set and let G: $X\times X\times X\to R^+$ be a function satisfying the following:
(G1) $G(x, y, z)=0$ if $x=y=z$,
(G2) $0< G(x, x, y)$ for all $x, y \in X$ with $x\neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$
for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z)=G(x, z, y)=G(y, z, x)=\ldots$ (symmetry in all variables),

(G5) $G(x, y, z)\leq G(x, a, a) +G(a, y, z)$ for all $x, y, z, a \in X$ (rectangular inequality),

Then the function G is called a generalised metric and the pair (X, G) is known as generalised metric spaces.

Now we introduce the notion of multiplicative generalised metric spaces similar to notion of multiplicative metric spaces defined by Bashirov [1] as follows:

Definition 1.1. Let X be a non empty set and let G: $X\times X\times X\to R^+$ be a function satisfying;
(MG1) $G(x, y, z)=1$ if $x=y=z$,
(MG2) $1 < G(x, x, y)$ for all $x, y \in X$ with $x\neq y$,
(MG3) $G(x, x, y) \leq G(x, y, z)$
for all $x, y, z \in X$ with $z \neq y$,
(MG4) $G(x, y, z)=G(x, z, y)=G(y, z, x)= \ldots$ (symmetry in all variables),
(MG5) $G(x, y, z)\leq G(x, a, a) .G(a, y, z)$ for all $x, y, z, a \in X$ (rectangular multiplicative inequality),

Then the function G is called a multiplicative generalised metric and we call the pair (X, G) as multiplicative generalised metric spaces. We shall denote it for briefly multiplicative G-metric spaces.

Example 1.1. Let $X =R^+$. Define $G :\times X\times X\to[1,\infty)$ as

$$G(x,y,z) = \left|\frac{x}{y}\right|^* . \left|\frac{y}{z}\right|^* . \left|\frac{z}{x}\right|^*, \text{where}$$

$$|x|^* = \begin{cases} x \text{ if } x \geq 1, \\ \frac{1}{x} \text{ if } x < 1. \end{cases} \text{ i.e., absolute multiplicative value.}$$

We note that property (MG1), (MG2), (MG3) and (MG4) holds clearly.

For(MG5),

we have $G(x,a,a).G(a,y,z) = \left|\frac{x}{a}\right|^* . \left|\frac{a}{a}\right|^* . \left|\frac{a}{x}\right|^* . \left|\frac{a}{x}\right|^* . \left|\frac{x}{y}\right|^* . \left|\frac{y}{a}\right|^*$

$$= \left|\frac{x}{a}\right|^* . \left|\frac{a}{x}\right|^* . \left|\frac{a}{y}\right|^* . \left|\frac{y}{z}\right|^* . \left|\frac{z}{a}\right|^*$$

$$> \left|\frac{x}{y}\right|^* . \left|\frac{y}{z}\right|^* . \left|\frac{z}{x}\right|^* = G(x,y,z) \text{ for all } x,y,z \in X.$$

Hence G is multiplicative generalised metric on X and (X, G) is multiplicative G-metric space.

Now we introduce some definition in setting of multiplicative G-metric space as follows

Definition 1.2. Let (X, G) be a multiplicative G-metric space The sequence $\{x_n\}$ in X is called a multiplicative G-Cauchy sequence if it holds that for all $\epsilon > 1$, there exists N $\in \mathbb{N}$ such that

$G(x_m, x_n, x_l) < \epsilon$ for all m, n, $l \geq$ N.

In 1986, Jungck [4] introduce the concept of compatible mappings. Now we define compatible mappings in this sapce as follows:

Definition 1.3. Two self mappings f and g of a multiplicative G-metric sapce are said to be compatible if

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 1$$

$or\ \lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 1$, whenever $\{x_n\}$ be a sequence I X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \text{ for some } t \in X.$

In 1996, Jungck and Rhodes[3] introduced the notion of weakly compatible maps as follows:

Definition 1.4. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Now we define expansive maps in multiplicative generalised metric space as follows:

Definition 1.5. Let (X, G) be multiplicative generalised metric space and T be a self mapping on X. Then T is said to be expansive mapping if there exists a constant a > 1 such that for all x, y, z ∈ X, we have

$$G^{\frac{1}{a}}(Tx, Ty, Tz) \geq G(x, y, z). (1.1)$$

Example 1.2. Let X be set of positive real numbers. Let us define G:X×X×X→[1, ∞) as

$$G(x,y,z) = \left|\frac{x}{y}\right|^* . \left|\frac{y}{z}\right|^* . \left|\frac{z}{x}\right|^*, \text{where}$$

$$|x|^* = \begin{cases} x \text{ if } x \geq 1, \\ \frac{1}{x} \text{ if } x < 1. \end{cases} \text{ i.e., absolute multiplicative value.}$$

Then clearly (X, G) be multiplicative G-metric space. Let us define a self map T on (x, G) as

T(x)=x^2. Then clearly for some a=2 > 1, (1.1) holds showing that T is an expansive map.

Many authors has proved fixed point results for expansive maps in different spaces [2,5].

2. Main Result

Now we prove fixed point results on expansive maps in multiplicative G-metric space as follows :

Theorem 2.1. Let (X, G) be a complete multiplicative G-metric space. If there exists a constant a > 1 and a surjective self map T on X such that for all x, y, z ∈ X

$$G^{\frac{1}{a}}(Tx, Ty, Tz) \geq G(x, y, z). (2.1)$$

Then T has a unique fixed point.

Proof. If Tx=Ty, then

$1 = G^{\frac{1}{a}}(Tx, Ty, Ty) \geq G(x, y, y)$,

which implies that G(x, y, y)=1 and hence x=y. So T is injective.

Consider the point $x_0 \in X$ then there exists $x_1 \in X$ such that T$x_1 = x_0$, Again $x_1 \in$ X then there exists $x_2 \in$ X such that T$x_2 = x_1$.

Continuing like this, we get sequence $\{x_n\}$ in X such that T$x_{n+1} = x_n$.

Now consider

$G(x_{n+1}, x_n, x_n) \leq G^{\frac{1}{a}}(Tx_{n+1}, Tx_n, Tx_n)$

$\leq G^{\frac{1}{a}}(x_n, x_{n-1}, x_{n-1})$.

Similarly we get $G(x_{n+1}, x_n, x_n) \leq G^{(1/a)^n}(x_1, x_0, x_0)$.

Consider $\frac{1}{a} = \lambda$.

Let m, n, l ∈ N such that m > n > l then we have

$$G(x_m, x_n, x_l) < G(x_m, x_{m-1}, x_{m-1}).G(x_{m-1}, x_n, x_l)$$
$$< G(x_m, x_{m-1}, x_{m-1}).G(x_{m-1}, x_{m-2}, x_{m-2}).$$
$$G(x_{m-2}, x_n, x_l)$$
$$< G(x_m, x_{m-1}, x_{m-1}) \ldots G(x_n, x_{n-1}, x_{n-1}) \ldots$$
$$G(x_{l+1}, x_l, x_l)$$
$$\leq (G(x_1, x_0, x_0))^{\lambda^{m-1}+\cdots+\lambda^l}$$
$$\leq (G(x_1, x_0, x_0))^{\frac{\lambda^l}{1-\lambda}}.$$

This implies that $G(x_m, x_n, x_l) \to 1\ as\ m, n, l \to \infty$. Hence the sequence $\{x_n\}$=$\{Tx_{n+1}\}$ is multiplicative G-Cauchy sequence. Since X is complete, there exists z ∈ X such that $x_n \to z$ as n→∞. Since T is surjective so there exists p ∈ X such that Tp=z.

Also

$G(z, Tp, x_n) \leq G^{\lambda}(Tz, Tz, Tx_n)$

$\leq G^{\lambda}(Tz, Tx_n, Tx_n) G^{\lambda}(Tx_n, Tz, Tx_n)$

Letting $n \to \infty$, *we get*

i.e.,G(Tz, Tz, z)=1. Therefore Tz=z. Hence z is a fixed point of T i.e., Tz=z.

Now if there is another point y(≠z) such that Ty=y then

G(z, y, y)=G(Tz, Ty, Ty) $\leq G^{\lambda}(z, y, y)$. Hence G(z, y, y)=1 and y=z, a contradiction. This implies that z is the unique fixed point of T.

Theorem 2.2. Let (X, G) be a complete multiplicative G-metric space. Let f and g be compatible self maps of X satisfying the condition

$$G^{\frac{1}{q}}(fx, fy, fz) \geq G(gx, gy, gz) \quad (2.2)$$

where q > 1 and g(X) ⊆ f(X), f is continuous. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, since g(X) ⊆ f(X), choose $x_1 \in X$ such that f x_1=gx_0. In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$. Using condition (2.2), consider

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq G^{\frac{1}{q}}(fx_n, fx_{n+1}, fx_{n+1}).$$

Which implies that $\{gx_n\}$ is Cauchy sequence. Since (X, G) is complete. Hence

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z.$$

Since f and g are compatible and f is continuous. So

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = fz.$$

Consider

consider $G(gfx_n, gfx_n, gz) \leq G^{\frac{1}{q}}(ffx_n, ffx_n, fz)$.

Proceeding limit n→∞, we get

$\lim_{n\to\infty} gfx_n = gz = fz.$ Now to show z is fixed point of f and g.

consider $G(gz, gx_n, gx_n) \leq G^{\frac{1}{q}}(fz, fx_n, fx_n)$.

Letting limit n→∞, we get

$$G(gz, z, z) \leq G^{\frac{1}{q}}(fz, z, z).$$

which implies that fz=z. Hence z is fixed point of f and g.

Uniqueness follows easily from (2.2).

Theorem 2.3. Let (X, G) be a complete multiplicative G-metric space. Let f and g be weakly compatible self maps of X satisfying the condition (2.2) and g(X) ⊆ f(X), if one of the subspaces g(X) or f(X) is complete. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, since g(X) ⊆ f(X), choose $x_1 \in X$ such that f x_1=gx_0. In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$. Using condition (2.2), consider

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq G^{\frac{1}{q}}(fx_n, fx_{n+1}, fx_{n+1}).$$

Which implies that $\{gx_n\}$ is Cauchy sequence. Since (X, G) is complete. Hence

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z.$$

Since f(X) is complete. So there exists a point p ∈ X such that fp=z.

Using (2.2)

$$G(gp, gx_n, gx_n) \leq G^{\frac{1}{q}}(fp, fx_n, fx_n).$$

Proceeding limit n→∞, we have

$$G(gp, z, z) \leq G^{\frac{1}{q}}(fp, z, z), \text{which implies that}$$

gp = z.

Therefore gp=fp=z. Since f and g are weakly compatible, therefore

fgp=gfp, i.e., fz=gz.
Now we claim that z is fixed point of f and g.

consider $G(gz, gx_n, gx_n) \leq G^{\frac{1}{q}}(fz, fx_n, fx_n)$.

Letting limit n→∞, we get

$$G(gz, z, z) \leq G^{\frac{1}{q}}(fz, z, z).$$

which implies that fz=z. Hence z is fixed point of f and g.
Uniqeness follows easily from (2.2).

Theorem 2.4. Let (X, G) be a multiplicative G-metric space. Let f and g be weakly compatible self maps of X satisfying the condition (2.2) and g(X) ⊆ f(X), if one of the subspaces g(X) or f(X) is complete. Then f and g have a unique common fixed point.

Proof. From the proof of Theorem (2.3), we conclude that $\{y_n\}$ is a Cauchy sequence in X. Now suppose that f(X)is complete subspace of X then there exists u, v ∈ X such that $\{y_n\}$ converges to v=fu.

Using (2.2)

$$\mathrm{G}(\mathrm{gu}, \mathrm{g}x_n, \mathrm{g}x_n) \leq G^{\frac{1}{q}}(\mathrm{fu}, \mathrm{f}x_n, \mathrm{f}x_n).$$

Letting n→∞, we get fu=gu=v, which shows that the pair f and g has a coincidence point. Since f and g are weakly compatible, therefore

fgu=gfu i.e., fv=gv.

Now we show that v is fixed point of f and g.

Conside $\mathrm{G}(\mathrm{gv}, \mathrm{g}x_n, \mathrm{g}x_n) \leq G^{\frac{1}{q}}(\mathrm{fv}, \mathrm{f}x_n, \mathrm{f}x_n)$.

Proceeding limit n→∞, we get

$$\mathrm{G}(\mathrm{gv}, v, v) \leq G^{\frac{1}{q}}(\mathrm{fv}, v, v)$$

$which\ implies\ that\ fv = v = gv$. Hence v is common fixed point of f and g.
Uniqueness follows easily.

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